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Albert-Ludwigs-Universität Freiburg
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Model Theory

Sheet 1

Deadline: 23.10.25, 2:30 pm.

Exercise 1 (2 points)**.**

Let $\mathcal{B}$ be an $\mathcal{L}$-structure and $(\mathcal{A}_i)_{i\in I}$ a directed family of substructures of $\mathcal{B}$ with $\mathcal{A}_i \preceq \mathcal{B}$ for all i in I. Using Tarski's test, show that $\bigcup_{i\in I} \mathcal{A}_i \preceq \mathcal{B}$.

Definition: A consistent theory T is *model complete* if the following condition holds: Whenever $\mathcal{A}$ and $\mathcal{B}$ are models of T such that $\mathcal{A}$ is a substructure of $\mathcal{B}$, then $A \preceq \mathcal{B}$.

Exercise 2 (7 points)**.**

Consider the language $\mathcal{L}$ consisting of a single binary relation symbol E. Let $\mathcal{K}$ be the class of $\mathcal{L}$-structures $\mathcal{A}$ such that the interpretation of the relation E is an equivalence relation with infinitely many equivalence classes. Furthermore, each $E^{\mathcal{A}}$-equivalence class has at most 2 elements.

a) Provide an axiomatization T.

b) Is T consistent? Is T complete?

c) Let $\mathcal{A}$ be a countable model of T with exactly one equivalence class of size 1, and let $\mathcal{B}$ be a countable model of T with exactly two equivalence classes of size 1. Show that $\mathcal{A}$ can be embedded into $\mathcal{B}$.

d) Is T model complete?

Exercise 3 (8 points)**.**

Consider a model $\mathcal{A}$ of the (consistent) $\mathcal{L}$-theory T.

a) Show that T is model complete if T has quantifier elimination.

b) Show that an $\mathcal{L}_A$-structure $\mathcal{C}$ is a model of $\mathrm{Diag}^{\mathrm{at}}(\mathcal{A})$ if and only if there is an $\mathcal{L}$-embedding $f : \mathcal{A} \to \mathcal{C}$.

c) Assume that T is model complete. Show that the $\mathcal{L}_A$-theory $T \cup \mathrm{Diag}^{\mathrm{at}}(\mathcal{A})$ is complete.

d) Suppose now that for every $\mathcal{L}$-formula $\varphi[x_1, \ldots, x_n]$, there exists a universal formula

$$\psi[x_1, \ldots, x_n] = \forall y_1 \ldots \forall y_m\, \theta[x_1, \ldots, x_n, y_1, \ldots, y_m] \text{ with } \theta \text{ quantifier-free}$$

such that $T \models \forall \bar{x}(\varphi[\bar{x}] \leftrightarrow \psi[\bar{x}])$. Show that T is model-complete.

Hint: Sheet 0, exercise 1.

(Please turn the page!)

Exercise 4. (3 points)

Consider the language $\mathcal{L} = \{0, +, -, (\lambda_q)_{q \in \mathbb{Q}}\}$ and let T be the $\mathcal{L}$-theory of $\mathbb{Q}$-vector spaces. Show that there is no $\mathcal{L}$-formula $\varphi[x, y]$ such that in every $\mathbb{Q}$-vectorspace V (viewed as an $\mathcal{L}$-structure $\mathcal{V}$) the following equivalence holds for all v and w in V:

$$\mathcal{V} \models \varphi[v, w] \quad \Leftrightarrow \quad v \text{ and } w \text{ are linearly dependent.}$$

In particular, the collection of pairs of linearly dependent vectors is not definable.

Hint: Sheet 0, exercise 2.

THE EXERCISE SHEETS CAN BE HANDED IN IN PAIRS. SUBMIT THEM IN THE MAILBOX 3.19 IN THE BASEMENT OF THE MATHEMATICAL INSTITUTE.